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ON THE PRIMARITY OF H^∞-SPACES

BY
J. BOURGAIN

ABSTRACT
It is proved that the poly-disc H^∞ spaces $H^\infty(D^m)$ are primary. We use the representation of these spaces as the direct sum of certain polynomial spaces and some properties of the Fourier transforms of $L^1(\Pi^m)$ functions.

1. Introduction

In this note D denotes the open disc $\{z \in \mathbf{C}; |z| < 1\}$, Π the circle $\{z \in \mathbf{C}; |z| = 1\}$ equipped with Haar measure (denoted by m or $|\ |$) and $H^\infty(D^m)$ ($m = 1, 2, \cdots$) is the space of bounded analytic functions on D^m. This space identifies also with the subspace of $L^\infty(\Pi^m)$ of those functions with Fourier transform contained in $(\mathbf{Z} \setminus \mathbf{Z}_-)^m$, where $\mathbf{Z}_-$ are the strictly negative integers.

It was shown in [4] that $H^\infty(D^m)$ is isomorphic to the space $(\Sigma_{n=1}^\infty H^\infty(D^m))_\infty$, i.e. the direct sum in l^∞-sense. Our purpose is to prove the following fact, which will be an application of the latter result:

THEOREM. *If $H^\infty(D^m)$ decomposes as the direct sum of two spaces X, Y, then either X or Y is isomorphic to $H^\infty(D^m)$.*

This answers affirmatively a question considered in [2]. Since the argument is completely analogous for $m \geqq 2$, we only present the one-variable case for simplicity sake. In fact, P. Wojtaszczyk obtained more recently the isomorphism $H^\infty(B_m) \sim (\Sigma_{n=1}^\infty H^\infty(B_m))_\infty$ (B_m denoting the open unit ball in $\mathbf{C}^m$) and it is likely that the method explained below permits one to prove the above theorem for these spaces also.

Let $L_N^\infty = [1, e^{i\theta}, e^{2i\theta}, \cdots, e^{Ni\theta}]_\infty$ for $N = 0, 1, 2, \cdots$ be the space of polynomials on Π of degree $\leqq N$, equipped with L^∞-norm. In [1], the following isomorphism is proved (as a consequence of [4]):

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$$H^\infty \sim \left(\sum_{N=0}^{\infty} L_N^\infty \right)_\infty .$$

As expected, we will first show how to derive the theorem from its finite-dimensional version.

PROPOSITION. *Given a positive integer n, there exists an integer $N(n)$ such that if $N \geqq N(n)$ and T is a linear operator on L_N^∞, the identity on L_n^∞ factorizes (boundedly w.r.t. $\|T\|$) either through T or $I - T$. Thus there are operators U, V factorizing*

$$\begin{array}{ccc} L_n^\infty & \xrightarrow{\ \mathrm{Id}\ } & L_n^\infty \\ {\scriptstyle U}\downarrow & & \uparrow{\scriptstyle V} \\ L_N^\infty & \xrightarrow{\ T_1\ } & L_N^\infty \end{array}$$

where $T_1 = T$ or $T_1 = I - T$ and $\|U\|\,\|V\| \leqq C(\|T\| + 1)$.

To prove the proposition, we first reduce to the case where T is (almost) a multiplier and then settle this particular situation. Only this part of the proof is more delicate.

2. Reduction to the finite dimensional question

Denote L_N^∞ by X_N. We will use following fact:

LEMMA 1. *Given $n \in \mathbf{Z}_+$, $\varepsilon > 0$, there is $N(n, \varepsilon)$ such that if $N \geqq N(n, \varepsilon)$ and E an n-dimensional subspace of X_N, there is a subspace F of X_N and a projection Q from X_N onto F such that*

(1) *$d(F, X_p) = 1$ for some $p \geqq n$ (where d is the Banach–Mazur distance),*

(2) $\|Q\| = 1$,

(3) *$\|Qx\| \leqq \varepsilon \|x\|$ for $x \in E$.*

PROOF. F will be a space L_Λ^∞ where $\Lambda = \{0, 1, \cdots, N\} \cap (d\mathbf{Z} + r)$, for $d = [N/n]$ and some $r = 0, 1, \cdots, d - 1$. Let Q be the restriction to X_N of the coset projection.

For N large enough, it is possible to choose r so that (3) holds. Remark first that for fixed $x \in X_N$

$$\sum_{n=0}^{N} |\langle x, e^{in\theta} \rangle| \leqq \sqrt{N} \|x\|$$

and then average

$$\frac{1}{d}\sum_{r=0}^{d-1}\sum_{n\in\Lambda_r}|\langle x, e^{in\theta}\rangle| \leqq \frac{n}{\sqrt{N}}\|x\|.$$

It remains to represent the elements of E using some Auerbach basis to conclude.

REMARKS. A previous random argument will be used again later on. Since X_n factorizes boundedly through X_p for $p > 2n$, condition (1) (resp. (2)) of Lemma 1 can be replaced by $d(F, X_n) \leqq C$ (resp. $\|Q\| \leqq C$).

Our purpose is to deduce the theorem from the proposition. Let, more generally, T be an operator acting on $(\Sigma_{n=0}^{\infty} X_N)_\infty$.

LEMMA 2. *Given $\varepsilon > 0$, there exists an operator T' on $(\Sigma_{n=0}^{\infty} X_n)$ satisfying*

(1) T' (resp. $I - T'$) *factorizes through* T (resp. $I - T$),

(2) T' *is "almost" diagonal in the sense that* $\|T' - D\| < \varepsilon$, *where D on $(\Sigma_{n=0}^{\infty} X_n)_\infty$ satisfies $D = \bigoplus(P_n D P_n)$ (P_n = nth coordinate projection).*

PROOF. For each n the operators $P_n T'(P_1, \cdots, P_{n-1})$ and $P_n T'(P_{n+1}, P_{n+2}, \cdots)$ must have small norm. The first condition is realized using Lemma 1. For the second, just observe that if S is a finite rank operator on $(\Sigma X_N)_\infty$ and $\varepsilon > 0$, there is always an infinite subset I of $\mathbf{Z}_+$ so that $\|SP_I\| < \varepsilon$, denoting P_I the obvious projection. Details are standard and left to the reader.

Lemma 2 clearly reduces the factorization of the identity on $(\Sigma X_n)_\infty$ through T or $I - T$ to the case where T is diagonal. But in this case the result immediately follows from the proposition stated above.

3. Use of the Féjèr kernel

$$F_n(\theta) = \sum_{j=-n}^{n} \frac{n - |j|}{n} e^{ij\theta}$$

is the Féjèr kernel and F_n will also denote the corresponding convolution operator.

LEMMA 3. *The identity on L_n^∞ factorizes through the direct sum operator $F_n \bigoplus F_n$ acting on $L_n^\infty \bigoplus L_n^\infty$.*

PROOF. If we define $U : L_n^\infty \to L_n^\infty \bigoplus L_n^\infty$ by $U\alpha = (\alpha, e^{in\theta}\bar{\alpha})$ and $V : L_n^\infty \bigoplus L_n^\infty \to L_n^\infty$ by $V(\alpha, \beta) = \alpha + \bar{\beta} e^{in\theta}$, clearly

$$\mathrm{Id}_{L_n^\infty} = V \circ (F_n \bigoplus F_n) \circ U.$$

LEMMA 4. *Assume given $\varepsilon > 0$, $n \in \mathbf{Z}_+$, $N \geqq N(n, \varepsilon)$ and T an operator on L^∞_N. Then there exist operators T_1, T_2, T_3 on L^∞_n and an operator T' on $L^\infty_n \oplus L^\infty_n \oplus L^\infty_n$ satisfying*

(1) T' (resp. $I - T'$) *factorizes boundedly through* T (resp. $I - T$),

(2) $\| T_i \| < C \| T \|$ $(i = 1,2,3)$,

(3) $\| T' - (T_1 \oplus T_2 \oplus T_3) \| < \varepsilon$.

PROOF. Iteration of the argument considered in the proof of Lemma 1 yields disjoint subsets Λ_i $(i = 1,2,3)$ of $\{0,1,\cdots,N\}$ obtained by intersection with a suitable coset so that

(4) $|\Lambda_i| \geqq n$ $(i = 1,2,3)$,

(5) $\| P_{\Lambda_i} T P_{\Lambda_j} \| < \varepsilon/2$ for $i \neq j$ (P_Λ being the orthogonal projection).

Denote $L^\infty_{\Lambda_i}$ by S_i and let $j_i : S_i \to L^\infty_N$ be the injection. If $T_i = P_{\Lambda_i} T j_i$, $\| T_i \| \leqq \| T \|$. Define $U : S_1 \oplus S_2 \oplus S_3 \to L^\infty_N$ (resp. $V : L^\infty_N \to S_1 \oplus S_2 \oplus S_3$) by $U = j_1 + j_2 + j_3$ (resp. $V = P_{\Lambda_1} \oplus P_{\Lambda_2} \oplus P_{\Lambda_3}$) and let $T' = VTU$. Condition (3) easily follows from (5).

Thus to prove the proposition, it suffices to factorize F_n. One can then indeed apply Lemma 4 and factorize $F_n \oplus F_n$ through $T_1 \oplus T_2 \oplus T_3$ or $(I - T_1) \oplus (I - T_2) \oplus (I - T_3)$. Thus, by Lemma 3, $\mathrm{Id}_{L^\infty_n}$ can be factorized through T or $I - T$.

4. Reduction to the multiplier case

A multiplier on L^∞_Λ is the restriction of a convolution operator.

LEMMA 5. *Given $n \in \mathbf{Z}_+$, $\varepsilon > 0$ there exists $N(n, \varepsilon)$ such that if $N \geqq N(n, \varepsilon)$ and T an operator acting on L^∞_N ($\| T \|$ bounded by some fixed constant), there exist a subset Λ of $\{0,1,\cdots,N\}$ obtained by intersection with a coset, such that $\| P_\Lambda T |_{L^\infty_\Lambda} - T' \| < \varepsilon$, where T' is a multiplier on L^∞_Λ, and $|\Lambda| \geqq n$.*

PROOF. We use again an averaging argument, considering sets

$$\Lambda_{d,r} = \{0,1,\cdots,N\} \cap (d\mathbf{Z} + r)$$

where

$$(d,r) \in \mathcal{P} = \{(d,r); [N/n^2] < d < [N/n] \text{ and } r = 0,1,\cdots,d-1\}.$$

Thus we estimate

$$|\mathcal{P}|^{-1} \sum_{(d,r)\in\mathcal{P}} \sum_{\substack{m,n\in\Lambda_{d,r} \\ m\neq n}} |\langle Te^{im\theta}, e^{in\theta}\rangle|$$

which can be rewritten as

$$|\mathcal{P}|^{-1} \sum_{0 \leqq m \leqq N} \sum_{\substack{[Nn^{-2}] < d < [Nn^{-1}] \\ 0 \leqq jd+m \leqq N}} |\langle Te^{im\theta}, e^{i(jd+m)\theta} \rangle|.$$

Since for fixed m

$$\sum_k |\langle Te^{im\theta}, e^{ik\theta} \rangle| \leqq \|T\| N^{1/2}$$

we find a majoration of the order $N^{-1/2} n^4 \|T\|$. Hence, for N large enough, $P_{\Lambda_{d,r}} T|_{L^\infty_{\Lambda_{d,r}}}$ will be almost diagonal for some $(d,r) \in \mathcal{P}$.

The proof of the proposition is thus reduced to factorization of F_n through T or $I - T$, where T is a multiplier on L^∞_N, $N \geqq N(n)$ (the factorization norm must, of course, be controlled by $\|T\|$).

5. Existence of certain arithmetic progressions

The Fourier–Stieltjes transform of a function $f \in L^1(\Pi)$ is given by

$$\hat{f}(n) = \int f(\theta) e^{-in\theta} m(d\theta) \qquad (n \in \mathbf{Z}).$$

$A(\mathbf{Z})$ denotes as usual the algebra of those transforms and is equipped with the obvious norm. The purpose of this section is to prove the following fact:

LEMMA 6. *Given $\varepsilon > 0$, $D < \infty$, $n \in \mathbf{Z}_+$, there exists an integer $N = N(\varepsilon, B, n)$ such that for each $\xi \in A(\mathbf{Z})$, $\|\xi\| < B$, there is some arithmetic progression in* $\{0, 1, \cdots, N\}$

$$P : r < r + d < r + 2d < \cdots < r + nd$$

satisfying

(1) $r < d$,

(2) $|\xi_k - \xi_l| < \varepsilon$ *for* $k, l \in P$.

This lemma asserts an oscillation property for elements of $A(\mathbf{Z})$. The reader will easily convince himself that the result does not hold for members of $l^\infty(\mathbf{Z})$ in general and hence is not a consequence of Van der Waerden's theorem [3].

By convolution, we can assume $\xi_k = \hat{f}(k)$ for $k \in \{0, 1, \cdots, N\}$ where $f \in L^1(\Pi)$, $\|f\|_1 < 5B$ and Spec $f \subset [-2N, 2N]$. Define for $0 < \tau < 1$

$$f^\tau = f\chi_{[|f| > \tau N]} \quad \text{and} \quad f_\tau = f\chi_{[|f| \leqq \tau N]}.$$

It will suffice to exhibit a progression P satisfying (1) of Lemma 6 and $0<\kappa<\tau<1$ such that

(3) $\|f-f^{\tau}-f_{\kappa}\|_1<\varepsilon/6$,

(4) $|\hat{f}^{\tau}(k)-\hat{f}^{\tau}(1)|<\varepsilon/3$ for $k,l\in P$,

(5) $|\hat{f}_{\kappa}(k)|<\varepsilon/6$ for $k\in P$.

We will show that if τ is not too small w.r.t. N, it is possible to find a progression P and $0<\kappa<\tau$, $\kappa=\kappa(\tau,n,\varepsilon)$ such that (4), (5) hold. Iterating then this principle leads to a decreasing sequence

$$\tfrac{1}{2}=\kappa_0>\kappa_1>\kappa_2>\cdots$$

which (for N large enough) can be constructed long enough to yield (3) for some choice $\tau=\kappa_s$, $\kappa=\kappa_{s+1}$.

First, an integer d will be found so that

$$\rho N<d<\frac{N}{n}\qquad\text{with } \rho=\rho(\tau,n,\varepsilon)$$

and (4) is verified for each translation $P=Q+r$ where $Q=\{0,1,\cdots,n\}\cdot d$.

Then, it will be shown that for κ small enough (5) will hold for some $0\leqq r<d$.

Denote by K the De la Vallée Poussin kernel with transform $\hat{K}=1$ on $[-2N,2N]$, $\hat{K}=0$ outside $[-3N,3N]$. For $t\in\Pi$, let δ_t be the Dirac measure at the point t.

LEMMA 7. *For $\eta>0$ there are points $t_1,\cdots,t_b\in\Pi$ and scalars $(c_j)_{1\leqq j\leqq b}$ such that*

(1) $b\leqq b(\tau,\eta,B)$,

(2) $\Sigma|c_j|\leqq\|f\|_1$,

(3) $\|(f^{\tau}-\Sigma_{1\leqq j\leqq b}c_j\delta_{t_j})*K\|_1<\eta$.

PROOF. For $\theta\in\Pi$, take $K_\theta(\psi)=K(\theta+\psi)$. Partition Π in intervals I_a of length γN^{-1} where $\gamma>0$ will be defined later. Choose $\theta_a\in I_a$ for each a. Then

$$\left\|(f^{\tau}*K)-\sum_a\int_{I_a}f^{\tau}\cdot K_{\theta_a}\right\|_1\leqq\sum_a\int_{I_a}|f^{\tau}(\theta)|\,\|K_\theta-K_{\theta_a}\|_1\leqq CB\gamma$$

(where C denotes a numerical constant). For γ small enough, an estimation by η is obtained in the previous line. It remains to bound $|J|$ where $J=\{a\,;f^{\tau}\neq 0$ on $I_a\}$.

If $t\in I_a$ and $|f(t)|>\tau N$, then clearly (since $\operatorname{Spec}f\subset[-2N,2N]$)

$$\int_{I_a}|f|\geqq\gamma N^{-1}(|f(t)|-\gamma N^{-1}\|\nabla f\|_\infty)\geqq\gamma(\tau-10\gamma B)>\tfrac{1}{2}\gamma\tau$$

if $\gamma < \tau/20B$. Hence $|J| < 2(\gamma\tau)^{-1}\|f\|_1$, proving the lemma.

An entropy argument yields an integer d so that $\rho'(\tau,\eta,B)N < d < N/n$ and

$$|e^{idt_j} - 1| < \eta \qquad \text{for } j = 1,\cdots,b.$$

For k, l in any translation P of Q defined above

$$|\hat{f}^\tau(k) - \hat{f}^\tau(l)| \leqq 2\eta + \sum_{1\leqq j\leqq b} |c_j|\,|\hat{\delta}_{t_j}(k) - \hat{\delta}_{t_j}(l)| \leqq 2\eta + \|f\|_1 n\eta < \varepsilon/3$$

for appropriate choice of η.

To realize the second step, i.e. the choice of $0 \leqq r < d$ and κ, again an averaging argument can be applied:

$$\frac{1}{d}\sum_{0\leqq r<d}\ \sum_{k\in Q+r} |\hat{f}_\kappa(k)|^2 \leqq \frac{n}{d}\|f_\kappa\|_2^2 \leqq \frac{n}{d}\kappa N\|f\|_1 < 5n\rho^{-1}B\kappa.$$

Take $\kappa = \varepsilon^2\rho/100nB$. Then some $0 \leqq r < d$ can be found such that P fulfils (5).

6. End of the proof of the proposition

Assume thus T a multiplier on L^∞_N induced by some element $\xi \in A(\mathbf{Z})$ ($\|\xi\| < B$, B being some numerical constant). Fix $n \in \mathbf{Z}_+$, $\varepsilon > 0$ and (for N large enough) apply Lemma 6 to obtain the progression P. Let us consider the following operators:

$$U : L^\infty_n \to L^\infty_N, \qquad U\alpha = \sum_{0\leqq j\leqq n} \hat{\alpha}(j) e^{i(jd+r)\theta},$$

$$V : L^\infty_N \to L^\infty_P, \quad \text{obtained by convolution with } K, \qquad K(\theta) = F_n(e^{id\theta})e^{ir\theta},$$

$$W : L^\infty_P \to L^\infty_n, \qquad W\alpha = \alpha(\theta/d)e^{-ir\theta/d}.$$

Notice that (1) of Lemma 6 asserts that V ranges in L^∞_P. Define $T' = T$ or $T' = I - T$ in order to ensure $|\xi'_k - \sigma| < \varepsilon$ for $k \in P$, where $\sigma \in \mathbf{C}$, $|\sigma| \geqq \frac{1}{2}$. Clearly for $\alpha \in L^\infty_n$

$$(WVT'U)\alpha = \sum_{0\leqq j\leqq n} \hat{\alpha}(j)\xi'_{jd+r}\hat{F}_n(j)e^{ij\theta}$$

and

$$\|(WVT'U)\alpha - \sigma F_n(\alpha)\|_\infty \leqq (n+1)\varepsilon\,\|F_n(\alpha)\|_\infty.$$

Choosing $\varepsilon < 1/10n$, a standard perturbation argument yields a factorization of F_n through T'.

7. Remarks

(1) It is easily seen that the proof of the proposition stated in the introduction is specific for L^{∞}_{π}-spaces and does not work in the L^{p}_{π}-case ($p < \infty$).

(2) Lemma 6 has the following dual version for functions f on Π with absolutely converging Fourier series. If $f = \Sigma \hat{f}(j) e^{ij\theta}$, $\|f\|_{A(\Pi)} = \Sigma |\hat{f}(j)|$. A similar argument as used to obtain Lemma 6 shows the following fact:

There is a function $\sigma(\delta, n, \varepsilon)$ such that if $\|f\|_{A(\Pi)} \leqq 1$ and $I = [\theta_0, \theta_1]$ is an interval in Π, of length $\geqq \delta$, there exists a progression P in I

$$P = \{\theta, \theta + h, \cdots, \theta + nh\}$$

such that $|\theta - \theta_0| < |h|$, $|h| > \sigma$ and f has oscillation at most ε on P.

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Department of Mathematics
Vrije Universiteit Brussel
Pleinlaan 2, F7
1050 Brussels, Belgium